New insight on the
Odd-Even mass Staggering
and pairing correlation in drip-line nuclei

Chong Qi
Dept. of Physics, Royal Institute of Technology (KTH), Stockholm

Collaborators:
Sara Changizi, R. Wyss, R.J. Liotta (KTH, Stockholm)
A.N. Andreyev (York), M. Huyse, P. Van Duppen (KU Leuven, Belgium)
Outline

- Brief introduction to odd-even staggering (OES) in nuclear binding energy
- Exact solution of the pairing Hamiltonian and comparison with the BCS and Richardson approaches
- OES and the residual pairing correlation
- Alpha cluster formation amplitudes in heavy nuclei and their relation with the pairing collectivity;
- Summary
The Nuclear Landscape

Odd-even staggering (OES)

\[ \Delta_n = \frac{1}{2}[2B(N, Z) - B(N + 1, Z) - B(N - 1, Z)] \]
OES may be attributed to:

- Pairing correlation effect/pair energy
- Deformation effect/mean field effect

\[ \Delta_n = \frac{1}{2}[2B(N,Z) - B(N+1,Z) - B(N-1,Z)] \]

Examples of OES formulae

\[
\Delta^{(3)}(N) = -\frac{1}{2} \left[ B(N - 1, Z) + B(N + 1, Z) - 2B(N, Z) \right] \\
= -\frac{1}{2} \left[ S_n(N + 1, Z) - S_n(N, Z) \right] \\
= \frac{1}{2} \left[ S_n(N, Z) - S_n(N - 1, Z) \right] \\
= \Delta^{(3)}(N) + \Delta^{(3)}_C(N). \tag{1}
\]

\[
\Delta^{(4)}(N) = \frac{1}{4} \left[ -B(N + 1, Z) + 3B(N, Z) \\
- 3B(N - 1, Z) + B(N - 2, Z) \right] \\
= \frac{1}{2} \left[ \Delta^{(3)}(N) + \Delta^{(3)}_C(N) \right].
\]

\[
\Delta^{(5)}(N) = \frac{1}{8} \left[ B(N + 2, Z) - 4B(N + 1, Z) \\
+ 6B(N, Z) - 4B(N - 1, Z) + B(N - 2, Z) \right] \\
= \frac{1}{4} \left[ \Delta^{(3)}_C(N + 2) + 2\Delta^{(3)}(N) + \Delta^{(3)}_C(N) \right].
\]
Examples of OES formulae

\[ \Delta^{(3)}(N) = -\frac{1}{2} \left[ B(N - 1, Z) + B(N + 1, Z) - 2B(N, Z) \right] = -\frac{1}{2} \left[ S_n(N + 1, Z) - S_n(N, Z) \right] \quad (1) \]

\[ \Delta^{(4)}(N) = \frac{1}{4} \left[ -B(N + 1, Z) + 3B(N, Z) - 3B(N - 1, Z) + B(N - 2, Z) \right] = \frac{1}{2} [\Delta^{(3)}(N) + \Delta^{(3)}(N)]. \]

\[ \Delta^{(5)}(N) = \frac{1}{8} \left[ B(N + 2, Z) - 4B(N + 1, Z) + 6B(N, Z) - 4B(N - 1, Z) + B(N - 2, Z) \right] = \frac{1}{4} [\Delta^{(3)}(N + 2) + 2\Delta^{(3)}(N) + \Delta^{(3)}(N)]. \]
Wigner effect

We hope that $\Delta^{(3)}_C$ contains **minimal contribution from the mean field** and is ‘free’ from the Wigner effect.

$$
\Delta^{(3)}_C(N) = \frac{1}{2} [S_n(N, Z) - S_n(N - 1, Z)]
$$
$$
= \frac{1}{2} [B(N, Z) + B(N - 2, Z) - 2B(N - 1, Z)]
$$
$$
= \frac{1}{2} [S_{2n}(N, Z) - 2S_n(N - 1, Z)]
$$
$$
= \Delta^{(3)}(N - 1),
$$

Neutron gaps for $Z = N$ nuclei in comparison with their corresponding fitted curves/average behavior.
Influence of the symmetry energy

FIG. 3. (Color online) Evolution of the single-particle energies of the $1p_{3/2}$, $1p_{1/2}$, $0f_{5/2}$ and $0g_{9/2}$ orbitals in Ca isotopes as a function of the neutron number $N$ for calculations with the standard Woods-Saxon parameters and $\kappa_{SO} = \kappa$ (open sym-

\[ V = V_0 \left( 1 + \frac{4\kappa}{A} \mathbf{t} \cdot \mathbf{T}_d \right), \]
OES in semi-magic nuclei and comparison with the HFB calculations
Pairing in a single-$j$ shell

\[ \hat{V} = a + bt_1 \cdot t_2 + G P_0, \]

\[ E(j^n) = \frac{1}{4} n(n-1)G - \left[ \frac{1}{2} n \right](j+1)G \]

First term takes into account the Pauli principle effect. It gives a minor contribution to the three-point OES formulae.

Seniority coupling in semi-magic nuclei

1943 Racah
1949 Goeppert-Mayer

Neutron separation energies from Ca isotopes

Proton separation energies from $N=28$ isotones

I. Talmi, Simple models of complex nuclei (Harwood, Chur, Switzerland, 1993)
The ‘competing’ BCS scheme

\[ E_v = \sqrt{(\epsilon_v - \lambda)^2 + \Delta_v^2} \]

\[ \Delta = G \sum_i u_i v_i, \]

uv measures the correlation of the wave function/collectivity of the stat.

For a single-j shell

\[ \Delta = \frac{G}{2} \sqrt{n(2j + 1 - n)}, \]

BCS:

- Does not conserve particle number
- Collapse at closed shell
- May have problems when applied to drip line nuclei
Two neutron transfer

**Seniority**

\[
\langle N + 2, v, \alpha | P^\dagger | N, v, \alpha \rangle = \frac{1}{2} \sqrt{(2\Omega - N - v)(N - v + 2)},
\]

\[
\langle N - 2, v, \alpha | P | N, v, \alpha \rangle = \frac{1}{2} \sqrt{(N - v)(2\Omega - N - v + 2)}.
\]

**BCS**

\[
\langle \text{BCS} | P^\dagger | \text{BCS} \rangle = \frac{\Delta}{G}.
\]

\[
\Delta = G \sum_i u_i \nu_i,
\]
Exact Solution of the pairing Hamiltonian for systems with many shells

Richardson’s approach

\[ H_P = \sum_k \varepsilon_k n_k + g \sum_{k,k'} c_{k\uparrow} \Gamma c_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow} \]

\[ |\Psi\rangle = \prod_{\alpha=1}^{M} \Gamma_{\alpha}^{\dagger} |0\rangle, \quad \Gamma_{\alpha}^{\dagger} = \sum_k \frac{1}{2\varepsilon_k - E_{\alpha}} c_{k\uparrow}^{\dagger} c_{-k\downarrow}^{\dagger} \]

Richardson equation

\[ 1 + g \sum_{k=0} \frac{1}{2\varepsilon_k - E_{\alpha}} + 2g \sum_{\beta(\neq \alpha)=1}^{M} \frac{1}{E_{\alpha} - E_{\beta}} = 0, \quad E = \sum_{\alpha=1}^{M} E_{\alpha} \]

Richardson equation

• A set of $M$ nonlinear coupled equations with $M$ unknowns ($E_\alpha$) and it is very difficult to solve.

• The pair energies are either real or complex conjugated pairs and do not have clear physical meaning.

• The wave function is not given directly.

\[
1 + g \sum_{k=0}^{M} \frac{1}{2\epsilon_k - E_\alpha} + 2g \sum_{\beta(\neq \alpha)=1}^{M} \frac{1}{E_\alpha - E_\beta} = 0, \quad E = \sum_{\alpha=1}^{M} E_\alpha
\]

For two pairs in a single-$j$ shell

\[
E_\alpha = -(\Omega - 1)g \pm i\sqrt{\Omega - 1}g
\]
Seniority coupling for many shells

- Shell model calculations restricted to the $v=0$ subspace
- There are as many independent solutions as states in the $v=0$ space.
- Valid for any forms of pairing.

\[
\langle \{s_j\}, \ldots N_j + 2, \ldots N_{j'} - 2, \ldots | H | \{s_j\}, \ldots N_j, \ldots N_{j'}, \ldots \rangle = \frac{G_{j,j'}}{4} \left[ (N_{j'} - s_{j'}) (2 \Omega_{j'} - s_{j'} - N_{j'} + 2) \right. \\
\left. \times (2 \Omega_j - s_j - N_j) (N_j - s_j + 2) \right]^{1/2}.
\]
What is the pairing correlation energy?

If one removes the self-energy, which may have been taken into account by the mean field, the binding energy of a single $j$ system can be rewritten as

$$E\left(j^n\right) = \frac{1}{4} n(n-1)G - \left[\frac{1}{2n}\right](j+1)G + \left[\frac{1}{2n}\right]G$$

For two particles in a non-degenerate system with a constant pairing, the energy can be evaluated through the well known relation,

\[
G \sum_i \frac{2j_i + 1}{2\varepsilon_i - E_2} = 2. \tag{10}
\]

The corresponding wave function amplitudes are given by

\[
X_i = N_n \frac{2j + 1}{2\varepsilon_i - E_2} \tag{11}
\]

The correlation energy induced by the monopole pairing corresponds to the difference

\[
\Delta = \varepsilon_\delta - \frac{1}{2}E_2, \tag{12}
\]

where \(\delta\) denotes the lowest orbital. As the gap \(\Delta\) increases the amplitude \(X_i\) becomes more dispersed, resulting in stronger two-particle correlation.
Seniority coupling involving many shells

- A way to solve the pairing Hamiltonian exactly
- Low-seniority configurations are dominant

FIG. 11. (Color Online) Solid symbol: The overlaps between the wave functions $|\Psi_i\rangle$ of the full Hamiltonian $H$ and those of the pairing Hamiltonian with $J_{\text{max}} = 0$ for the first $5/2^+$ and $7/2^+$ states in light odd-$A$ Sn isotopes. Open symbol: Same as above but only the non-diagonal pairing matrix elements are considered.
Binding energies of Ca isotopes

WS + constant pairing

FIG. 6. (Color online) Experimental [34, 37] and calculated ground-state energies of Ca isotopes, relative to that of $^{40}$Ca, as a function of mass number $A$.

A. T. Gallant et al., PRL 109, 032506 (2012)

Calculations with three-body interaction
Two-body clustering

Configuration mixing from higher lying orbits is important for clustering at the surface.

\[ \Psi_2(r_1, r_2) = (\chi_1\chi_2)0 \Phi_2(r_1, r_2, \theta_{12}) = (\chi_1\chi_2)0 \frac{1}{4\pi} \sum_p \sqrt{\frac{2j_p + 1}{2}} X_p \phi_p(r_1) \phi_p(r_2) P_{l_p}(\cos \theta_{12}), \]

\[ r_1 = 9 \text{ fm} \]

\( ^{210}\text{Pb} \)

\( ^{206}\text{Pb} \)

FIG. 10: (color online). The square of the two-neutron wave function \( |\Psi_{2\nu}(r_1, r_2, \theta)|^2 \) with \( r_1 = 9 \) fm as a function of \( r_2 \) and \( \theta \). Left: the leading configuration; Right: 4 major shells.
No Pairing

'Strong' pairing
Figure 5: Upper: The two-neutron correlation plots for $^{46}\text{Ca}$ (left) and $^{54}\text{Ca}$ (right). Lower: Same as upper but for $^{128}\text{Sn}$ (left, 4 holes) and $^{136}\text{Sn}$ (right, 4 particles). Notice that the scale is different.
Alpha formation probability from experiments

$$\log |RF(R)|^{-2} = \log T_{1/2}^{\text{Expt.}} - \log \left( \frac{\ln 2}{\nu} |H_0^+(\chi, \rho)|^2 \right),$$

$R$ should be large enough that the nuclear interaction is negligible, i.e., at the nuclear surface.

$$R = 1.2(A_d^{1/3} + A_c^{1/3})$$

210Po vs 212Po (The later is the textbook example of alpha emitter)

\[ |^{212}\text{Po}(\alpha_4)\rangle = \sum_{\alpha_2\beta_2} X(\alpha_2,\beta_2; \alpha_4) |^{210}\text{Pb}(\alpha_2) \otimes ^{210}\text{Po}(\beta_2)\rangle \]

If we neglect the proton-neutron interaction:

\[ |^{212}\text{Po}(\alpha, \text{g.s.})\rangle = |^{210}\text{Pb}(2\nu, \text{g.s.}) \otimes |^{210}\text{Po}(2\pi, \text{g.s.})\rangle, \]

\[ |^{210}\text{Po}(\alpha, \text{g.s.})\rangle = |^{206}\text{Pb}(2\nu^{-1}, \text{g.s.}) \otimes |^{210}\text{Po}(2\pi, \text{g.s.})\rangle. \]

\[ \mathcal{F}_\alpha(R; ^{212}\text{Po}(\text{gs})) = \int dR d\xi \phi_\alpha(\xi) \Psi(r_1r_2; ^{210}\text{Pb}(\text{gs})) \Psi(r_3r_4; ^{210}\text{Po}(\text{gs})), \]

\[ \mathcal{F}_\alpha(R; ^{210}\text{Po}(\text{gs})) = \int dR d\xi \phi_\alpha(\xi) \Psi(r_1r_2; ^{206}\text{Pb}(\text{gs})) \Psi(r_3r_4; ^{210}\text{Po}(\text{gs})). \]

Two-body clustering
The two-body wave functions are indeed strongly enhanced at the nuclear surface;
The enhancement is much weaker in $^{206}\text{Pb}(\text{gs})$ than that in $^{210}\text{Pb}(\text{gs})$

- Relatively small number of configurations in the hole-hole case;
- $p_{1/2}$ dominance in $^{206}\text{Pb}(\text{gs})$;
- Radial wave functions of hole states less extended.
\[ F_\alpha(R) = \sqrt{\frac{1}{4\pi}} \int dr_\alpha \phi_\alpha(r_\alpha) \Psi_{2\pi}(r_1, r_2) \Psi_{2\nu}(r_3, r_4), \]

- Alpha particle is formed on the nuclear surface;
- The clustering induced by the pairing mode is inhibited if the configuration space does not allow a proper manifestation of the pairing collectivity.

Pairing gap and the alpha formation

Cross sections of (p,t) reactions on Pb isotopes

M. Takahashi, PRC27, 1454(1983)
Proton decay

\[ \mathcal{F}_i(R) = \int dR d\xi_d [\Psi(\xi_d)\xi_p Y_i(R)]^* J_{m M_m} \Psi_m(\xi_d, \xi_p, R), \]

that \( \mathcal{F}_i(R) \) would indeed be the wave function of the outgoing particle \( \psi_p(R) \) if the mother nucleus would behave at the point \( R \) as

\[ \Psi_m(\xi_d, \xi_p, R) = [\Psi(\xi_d)\xi_p \psi_p(R) Y_i(R)] J_{m M_m}. \] (3)
FIG. 4. (Color online) The formation amplitudes $|F_l(R)|$ extracted from experimental data for proton decays of nuclei $N \geq 75$ ($Z > 67$) as a function of $u$ calculated from BCS calculations using for the Woods-Saxon mean field the universal parameters [23] (upper) and the Cherpunov parameters [24] (lower).
Summary

- OES and pairing gaps
- Seniority in many shells
- OES as an indication of the pairing collectivity
- Alpha clustering in heavy nuclei and pairing correlation
- Inclusion of continuum configurations
- Application in studying the pair correlation effects in decay and reaction processes

Thank you!
Microscopic description of alpha decay


\[
T_{1/2} = \frac{\hbar \ln 2}{\Gamma_c} = \frac{\ln 2}{\nu} \left| \frac{H_l^+(\chi, \rho)}{RF_c(R)} \right|^2,
\]

\( \nu \) is the outgoing velocity of the emitted particle
\( F_c(R) \) is the formation amplitude
\( H_l^+ \) is the Coulomb-Hankel function

The penetrability is proportional to \( |H_l^+(\chi, \rho)|^{-2} \).

\( R \) is the distance between the center of mass of the cluster and daughter nucleus which divides the decay process into an internal region and complementary external region.

\[ m \rightarrow d + \alpha \]

\[ F_l(R) = \int dR d\xi_d d\xi_\alpha [\Psi(\xi_d) \phi(\xi_\alpha) Y_l(R)]^* J_{m M_m} \Psi_m(\xi_d, \xi_\alpha, R), \]

Formation amplitude:
- Can be extracted from experimental data in a model-independent way;
- Can be calculated microscopically in a direct way;

Shell Model
H.J. Mang, PR 119,1069 (1960); I. Tonozuka, A. Arima, NPA 323, 45 (1979).

BCS approach
GN law works if the formation probability is a constant or proportional to $Q^{-1/2}$

$$\log|\text{RF}(R)|^2 \propto Q^{-1/2}$$
2p decay?

Strong pairing